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Naval Undersea Warfare Center Division
Newport, Rhode Island

ON THE RELATIONSHIP BETWEEN THE DISCRETE FOURIER TRANSFORM AND MAXIMUM LIKELIHOOD ESTIMATION

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This memorandum examines the relationship between the discrete Fourier transform (DFT) and Maximum Likelihood (ML) estimation when applied to the estimation of directions-of-arrival (DOAs). It is shown that the DFT/periodogram method, which can be used to obtain the maximum likelihood estimate for a single DOA, can be generalized to yield maximum likelihood estimates for multiple DOAs. A new DOA algorithm is presented and a fast implementation is developed. This algorithm can be modified to be as fast as and more accurate than conventional narrowband beamforming in many practical applications, with the ability to achieve the Cramer-Rao bound in situations requiring maximum resolution.

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1. INTRODUCTION

It is commonly known that for the estimation of the frequency of a single sinusoidal source in white Gaussian noise the method using a discrete Fourier transform (DFT) and periodogram is equivalent to maximum likelihood estimation. This method can also be used to find the ML estimate for a direction-of-arrival (DOA) corresponding to a single source in an array processing scenario. For multiple DOAs this method does not in general produce ML estimates. It will be demonstrated, however, that the DFT/periodogram estimation method can be generalized to provide maximum likelihood estimates for multiple sources.

This memorandum is organized as follows. In the second section the maximum likelihood equation and corresponding least-squares problem are given. In section 3 it is shown that the standard DFT/periodogram method for the ML estimation of a single DOA can be generalized to yield maximum likelihood estimates for multiple DOAs. A fast maximum likelihood estimation algorithm (FMLE) is discussed in section 4, and conclusions are given in section 5.

The DOA problem treated here arises from M narrowband farfield signals impinging on a linear array of N equally spaced sensors. The sensor outputs are subject to additive, zero-mean, white Gaussian noise. The equation describing this scenario is

$$y(k) = D(k)\rho(k) + \mu(k), \quad k = 1, 2, \ldots,$$

where $y(k)$, called a data snapshot, is an N-vector of array outputs at sampling time k. The columns of $D(k) = [d_1(k)\ d_2(k)\ \cdots\ d_M(k)]$ are steering vectors at time k, where

$$d_i = [1\,\exp(j\omega_i)\ \cdots\ \exp(j\omega_i(N-1))]^T, \quad i = 1, \ldots, M, \tag{1}$$

and $\omega_i \stackrel{\text{def}}{=} 2\pi\beta\cos(\theta_i)$. Here, θ_i is the i^{th} bearing and β is the interelement spacing of the array in wavelengths. The elements of the signal vector $\rho(k)$ are complex exponentials:

$$\rho_i(k) = c_i(k)\exp(j\varphi_i(k)), \ i = 1, \cdots, M, \ \ k = 1, 2, \ldots,$$

representing real signal amplitudes c_i and phases φ_i. The noise vector at time k is $\mu(k)$.

2. DEVELOPMENT OF THE LEAST-SQUARES PROBLEM

An $N-$vector of the form:

$$\left[1\ z\ z^2\ \cdots\ z^{N-1}\right]^T, \quad z = e^{j\theta}, \ j \stackrel{\text{def}}{=} \sqrt{-1},$$

will be called a *Fourier vector*. When Fourier vectors are mentioned in the context of array processing as in (1), they will also be referred to as *steering vectors.*

Treating the array processing problem in the frequency domain, consider a data snapshot represented by an N-vector consisting of samples from the elements of a linear array, where the data y_n from the n^{th} element consist of a weighted sum of M complex exponentials corresponding to M sources:

$$y_n = \sum_{i=1}^{M} \rho_i \exp(j\omega_i n) + \eta(n), \quad n = 1, \cdots, N,$$

where $\eta(n)$ is the n^{th} element of the noise vector, μ. The maximum likelihood estimates for ρ_i and ω_i involve the minimization of the following least-squares error:[1]

$$E = \sum_{n=1}^{N} \left| y_n - \sum_{i=1}^{M} \hat{\rho}_i \exp(j\hat{\omega}_i n) \right|^2, \tag{2}$$

where the $\hat{\rho}_i's$ and $\hat{\omega}_i's$ are estimates. Let $A = [y_1 \cdots y_N]^T$ and $D = [d_1 \ \cdots \ d_M]$, with $\rho = [\hat{\rho}_1 \ \cdots \hat{\rho}_M]^T$. The least-squares problem in (2) when generalized for r data snapshots can be written in matrix/vector form and stated as follows:

The M-Source Problem *Let A be a complex $N \times r$ matrix. Find an $N \times M$ matrix D with distinct Fourier vectors as columns and an $M \times r$ matrix ρ such that the following holds:*

$$\{D, \rho\} = \arg\min_{D,\rho} \ \|A - D\rho\|_F^2. \tag{3}$$

The matrix norm $\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2$ where $\|A\|_F$ denotes the Frobenius norm of the matrix A. Here, $\arg\min_{D,\rho}(\cdot)$ denotes the values of the matrices D and ρ, which minimize the given expression in parentheses, and similarly for $\arg\max_{D,\rho}(\cdot)$.

Using the projection theorem[2] the error vector $A - D\rho$ is orthogonal to the column space in which the solution lies. Therefore, the orthogonal projection of the error onto the column space of D will be zero. If the columns of D are assumed to be distinct, then the orthogonal projection matrix can be written as

$$D(D^H D)^{-1} D^H,$$

and thus

$$D(D^H D)^{-1} D^H (A - D\rho) = 0.$$

This can then be solved for ρ to obtain

$$\rho = (D^H D)^{-1} D^H A, \tag{4}$$

and it follows that

$$\arg\min_{D,\rho} \ \|A - D\rho\|_F^2 \ = \ \arg\max_{D} \ \text{trace} \left(A^H D(D^H D)^{-1} D^H A \right). \tag{5}$$

The formulation of the maximum likelihood problem in (5) can be found in Kay.[3] The remainder of this memorandum presents a method for improving the efficiency of the numerical calculation of (5).

3. EXTENSION OF THE DFT/PERIODOGRAM METHOD TO ML ESTIMATION OF MULTIPLE DOAS

In general the matrix A will have r columns, each corresponding to a data snapshot and the steering vector matrix D will have M columns representing an M-source model. A specific case will now be considered, where $r = 2$ and $M = 2$. Consider the following two source, two data snapshot problem:

$$\min_{i,j,\rho} \|A - \hat{D}_{ij}\rho\|_F^2, \tag{6}$$

where A is $N \times 2$, $\hat{D}_{ij}$ is $N \times 2$, and ρ is 2×2, and where $\hat{D}_{ij}$ and ρ are to be determined. Let $\mathcal{F}_N$ denote the DFT matrix of size N, scaled so that $\mathcal{F}_N$ is unitary: $\mathcal{F}_N\mathcal{F}_N^H = I_N$. Let the columns of D_{ij} be general Fourier vectors and let the columns of $\hat{D}_{ij}$ be Fourier vectors that are also columns of an inverse DFT matrix. (The fact that a vector f is a Fourier vector doesn't imply that it is a column of an inverse DFT matrix.) Let e_i be the unit vector with a $'1'$ in the i^{th} position and let $\mathcal{E}_{ij} \stackrel{\text{def}}{=} [e_i\ e_j]$.

Since the norm $\|\cdot\|_F$ is unitarily invariant[4] and $\mathcal{F}_N$ is a unitary matrix (6) is equivalent to

$$\min_{i,j,\rho} \|\mathcal{F}_N A - \mathcal{F}_N \hat{D}_{ij}\rho\|_F^2. \tag{7}$$

Let $C \stackrel{\text{def}}{=} \mathcal{F}_N A$. Since $\mathcal{F}_N \hat{D}_{ij} = \mathcal{E}_{ij}$, (7) is equivalent to

$$\min_{i,j,\rho} \|C - \mathcal{E}_{ij}\rho\|_F^2. \tag{8}$$

Here,

$$\mathcal{E}_{ij}\rho = \left(\begin{array}{cc} 0 & 0 \\ \vdots & \vdots \\ \rho_{11} & \rho_{12} \\ \vdots & \vdots \\ \rho_{21} & \rho_{22} \\ \vdots & \vdots \\ 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ \textbf{row i} \\ \\ \textbf{row j} \\ \\ \\ \end{array} .$$

Let

$$\mathbf{C}_{ij} \stackrel{\text{def}}{=} \begin{pmatrix} C_{i1} & C_{i2} \\ C_{j1} & C_{j2} \end{pmatrix}.$$

Given i and j, (8) is minimized by letting $\rho = \mathbf{C}_{ij}$, and so

$$\min_{i,j,\rho} \|C - \mathcal{E}_{ij}\rho\|_F^2 = \|C\|_F^2 - \max_{i,j} \|\mathbf{C}_{ij}\|_F^2. \tag{9}$$

This solution is in general of limited value since the search is performed over only a fixed number N of Fourier vectors, which are restricted to be columns of $\mathcal{F}_N^{-1}$. A finer angular increment than $2\pi/N$ is often necessary. To this end, a generalization of (6) will now be developed. Let the matrix F be the first N rows of $\mathcal{F}_L^{-1}$, with $L > N$. An angular increment $2\pi/L$ between sucessive Fourier N-vectors can be obtained by choosing them to be the columns of F. The minimization problem is now

$$\min_{i,j,\rho} \|A - D_{ij}\rho\|_F^2. \tag{10}$$

This is similar to (6) except that here D_{ij} has non-orthonormal Fourier N-vectors as columns. Consider the expression

$$\left\| \begin{pmatrix} A \\ 0 \end{pmatrix} - \hat{D}_{ij}\rho \right\|_F^2, \tag{11}$$

where the columns of the matrix A have been zero-padded to length L, and D_{ij} has been extended to $\hat{D}_{ij}$, where $\hat{D}_{ij}$ is now columns i and j of the matrix $\mathcal{F}_L^{-1}$. The expression in (11) can be rewritten as

$$\begin{aligned} & \left\| \begin{pmatrix} A \\ 0 \end{pmatrix} - \hat{D}_{ij}\rho \right\|_F^2 \\ = \ & \|A - D_{ij}\rho\|_F^2 - \|D_{ij}\rho\|_F^2 + \|\hat{D}_{ij}\rho\|_F^2 \\ = \ & \|A - D_{ij}\rho\|_F^2 - \|D_{ij}\rho\|_F^2 + \|\mathcal{F}_L\hat{D}_{ij}\rho\|_F^2 \\ = \ & \|A - D_{ij}\rho\|_F^2 - \|D_{ij}\rho\|_F^2 + \|\rho\|_F^2. \end{aligned} \tag{12}$$

Let B be defined as

$$B \stackrel{\text{def}}{=} \mathcal{F}_L \begin{pmatrix} A \\ 0 \end{pmatrix},$$

and $\mathcal{B}_{ij}$ as

$$\mathcal{B}_{ij} \stackrel{\text{def}}{=} \begin{pmatrix} B_{i1} & B_{i2} \\ B_{j1} & B_{j2} \end{pmatrix}.$$

An alternative expression for that in (11) is

$$\begin{aligned} & \left\| \begin{pmatrix} A \\ 0 \end{pmatrix} - \hat{D}_{ij}\rho \right\|_F^2 \\ = \ & \left\| \mathcal{F}_L \begin{pmatrix} A \\ 0 \end{pmatrix} - \mathcal{F}_L\hat{D}_{ij}\rho \right\|_F^2 \\ = \ & \|B - \mathcal{E}_{ij}\rho\|_F^2 \\ = \ & \|B\|_F^2 - \|\mathcal{B}_{ij}\|_F^2 + \|\mathcal{B}_{ij} - \rho\|_F^2 \\ = \ & \|B\|_F^2 - \text{ trace } (\mathcal{B}_{ij}^H\rho) - \text{ trace } (\rho^H\mathcal{B}_{ij}) + \|\rho\|_F^2. \end{aligned} \tag{13}$$

Equating (12) and (13):

$$\begin{aligned}\|A - D_{ij}\rho\|_F^2 &= \|B\|_F^2 + \|D_{ij}\rho\|_F^2 - \text{ trace } (\mathcal{B}_{ij}^H \rho) - \text{ trace } (\rho^H \mathcal{B}_{ij}) \qquad (14)\\ &= \|B\|_F^2 + \text{ trace } (\rho^H D_{ij}^H D_{ij} \rho) - \text{ trace } (\mathcal{B}_{ij}^H \rho) - \text{ trace } (\rho^H \mathcal{B}_{ij}).\end{aligned}$$

Given i and j, the value of ρ needed to minimize the expression in (14) can be obtained by differentiation. To this end, the following can be shown to hold whenever X and R are real matrices with appropriate dimensions:[5]

- $\frac{\partial}{\partial X}$ trace $(RX) = R^T$.
- $\frac{\partial}{\partial X}$ trace $(X^T R) = R$.

Noting that $\partial \overline{f(z)}/\partial z = 0$ [6] whenever $f(z)$ is an analytic function of a complex variable z, it follows that $\partial(X^H)/\partial X = 0$ for matrices X and X^H whose elements are independent complex variables. (The two variables z and $\bar{z}$ are considered to be independent since $\partial \bar{z}/\partial z = 0$.) The complex versions of the above formulas follow, where now R is a constant complex matrix and the elements of the matrices X and X^H are independent complex variables:

- $\frac{\partial}{\partial X}$ trace $(RX) = R^T$.
- $\frac{\partial}{\partial X}$ trace $(X^H R) = 0$.

Differentiating (14) with respect to ρ yields the following expression:

$$D_{ij}^T \bar{D}_{ij} \bar{\rho} - \bar{\mathcal{B}}_{ij}.$$

Conjugating, setting this expression equal to zero, and assuming that the columns of D_{ij} are distinct, one obtains

$$\rho = (D_{ij}^H D_{ij})^{-1} \mathcal{B}_{ij}. \qquad (15)$$

Substituting this into (14) gives the following result:

$$\arg\min_{i,j,\rho} \|A - D_{ij}\rho\|_F^2 = \arg\max_{i,j} \text{ trace } \left(\mathcal{B}_{ij}^H (D_{ij}^H D_{ij})^{-1} \mathcal{B}_{ij}\right). \qquad (16)$$

Let there be M DOAs to be estimated and assume that A has r columns. Define

$$\tilde{D}_\alpha \stackrel{\text{def}}{=} [d_{i_1} \ \cdots \ d_{i_r}]^T, \quad \alpha \stackrel{\text{def}}{=} i_1, \ldots, i_r.$$

Then, (16) generalizes to

$$\arg\min_{\alpha,\rho} \|A - \tilde{D}_\alpha \rho\|_F^2 = \arg\max_{\alpha} \text{ trace } \left(\mathcal{B}_\alpha^H (\tilde{D}_\alpha^H \tilde{D}_\alpha)^{-1} \mathcal{B}_\alpha\right), \qquad (17)$$

where $\mathcal{B}_\alpha$ is an $r \times p$ matrix of DFT coefficients. Equation (17) represents the major result of this paper. Comparing (17) with (5) it can be seen that the columns of the matrix of Fourier vectors D can be chosen in such a way that the matrix $A^H D$ in equation (5) can be replaced with the matrix $\mathcal{B}_\alpha$ in (17) whose elements consist of DFT coefficients. Thus,

with a proper choice of D all the inner products comprising the elements of $A^H D$ can been computed by calculating an FFT of each of the r columns of A.

When there are two sources and A is a single column vector, $\mathcal{B}_{ij}$ is a 2×1 matrix containing the i^{th} and j^{th} DFT coefficients of B. If A does not need to be zero-padded in this case in order to obtain maximum likelihood resolution, then $L = N$ and $D_{ij} = \hat{D}_{ij}$ so that D_{ij} has orthonormal columns and

$$\arg\min_{i,j,\rho} \|A - D_{ij}\rho\|_F^2 = \arg\max_{i,j} \text{ trace } \left(\mathcal{B}_{ij}^H \mathcal{B}_{ij}\right). \tag{18}$$

These are the values for i and j corresponding to the two largest magnitude values in the periodogram corresponding to the DFT of A. These can be found from a single one-dimensional search and the joint estimation of DOAs in this case is unnecessary. When one DOA is to be estimated from a single data snapshot, (18) reduces to

$$\arg\min_{i,\rho} \|A - D_i\rho\|_F^2 = \arg\max_{i} \left(\mathcal{B}_i^H \mathcal{B}_i\right).$$

As is well known, this selects the bearing corresponding to the maximum value of the periodogram, i.e., the DFT coefficient of largest magnitude, as the ML estimate for the bearing.

4. ALGORITHM IMPLEMENTATION

To implement the FMLE algorithm when there are two sources and a single data snapshot, the following search needs to be implemented efficiently:

$$\arg\max_{i,j} \text{ trace } \left(\mathcal{B}_{ij}^H (D_{ij}^H D_{ij})^{-1} \mathcal{B}_{ij}\right). \tag{19}$$

First it is shown that the elements of the matrix $(D_{ij}^H D_{ij})^{-1}$ can be computed in closed form. Consider two Fourier N-vectors d_i and d_j, where

$$d_i \stackrel{\text{def}}{=} \left[1 \exp(2\pi(\sqrt{-1}/N)i) \ \cdots \ \exp(2\pi(\sqrt{-1}/N)(N-1)i)\right]^T,$$

and similarly for d_j. Observing that $d_i^H d_j$ is the sum of the first N terms in a geometric series the following is obtained:

$$\gamma_{ij} \stackrel{\text{def}}{=} d_i^H d_j = \frac{1 - \exp(-(2\pi\sqrt{-1}/N)(i-j)N)}{1 - \exp(-(2\pi\sqrt{-1}/N)(i-j))}.$$

Given i and j, the matrix $D_{ij}^H D_{ij}$ can then be written as

$$D_{ij}^H D_{ij} = \begin{pmatrix} N & \gamma_{ij} \\ \bar{\gamma}_{ij} & N \end{pmatrix}.$$

and the inverse as

$$\left(D_{ij}^H D_{ij}\right)^{-1} = \frac{1}{N^2 - |\gamma_{ij}|^2}\begin{pmatrix} N & -\gamma_{ij} \\ -\bar{\gamma}_{ij} & N \end{pmatrix}.$$

Letting $\mathcal{B}_{ij} \stackrel{\text{def}}{=} [b_i \; b_j]^T$, (19) can then be expressed as

$$\arg\max_{i,j}\left\{\frac{N}{N^2 - |\gamma_{ij}|^2}\left(|b_i|^2 + |b_j|^2 - (2/N)\Re(\bar{b}_i b_j \gamma_{ij})\right)\right\}.$$

Writing $b_i = r_i e^{\sqrt{-1}\theta_i}$ and $\gamma_{ij} = R_{ij} e^{\sqrt{-1}\phi_{ij}}$, this can then be implemented in real arithmetic as

$$\arg\max_{i,j}\left\{\frac{N}{N^2 - R_{ij}^2}\left(r_i^2 + r_j^2 - (2/N) r_i r_j R_{ij}\cos(-\theta_i + \theta_j + \phi_{ij})\right)\right\}. \tag{20}$$

Equation (20) is of theoretical interest because it shows how to modify the value $r_i^2 + r_j^2$ of the point (i, j) in the two-dimensional periodogram based only on a single DFT of the data snapshot, in order to obtain a "normalized" periodogram whose maximum corresponds to the ML estimate. MATLAB code implementing (20) is given in the appendix. Note that the entries in $(D_{ij}^H D_{ij})^{-1}$ depend only on the difference $i - j$ of the indices. This is advantageous because, as can be seen in the appendix, the main loop can be over the difference $i - j$. Since $i - j$ is then constant for each iteration, the inverse matrix $(D_{ij}^H D_{ij})^{-1}$ is also constant and this allows the inner loop to be vectorized, with a resulting increase in efficiency, especially for small values of $i - j$.

The following example is intended to illustrate the use of the FMLE method to provide increased resolution in a bearing estimation scenario where Cramer-Rao (CR) bound performance is not a necessity. Assuming that there is an N element array and that a zero-padded FFT of $T \geq N$ points has been taken, the cost is approximately $8T^2$ flops, or floating-point operations. This is so because the preprocessing takes $O(T\log(T))$ flops for the FFT and $O(T)$ flops to precompute the magnitude, squared magnitude, and angular argument of each DFT coefficient. To implement beamforming, each beam takes N complex additions and multiplications, for a total of $8NT$ flops. The complexities of the two methods are then roughly equal when the number of beams is equal to the number of array elements. As T/N increases, the efficiency of the FMLE algorithm with respect to beamforming therefore decreases.

To improve the efficiency of the FMLE algorithm, first perform a coarse 2-D search to find two candidate DOAs p_1 and p_2. If these are separated by more than $1/N$, then ML estimates can be obtained by performing two fine 1-D searches over intervals $[a_1 \; a_2]$ and $[b_1 \; b_2]$ $(a_1 < b_1)$ with each interval centered on a candidate DOA. If p_1 and p_2 are not sufficiently separated $(a_2 + 1 \geq b_1)$ then perform a single fine 2-D search over the interval $[a_1 \; b_2]$. The two signal amplitudes and phases can be calculated according to (15), which gives maximum likelihood estimates:[3]

$$\rho_{ij} \stackrel{\text{def}}{=} (D_{ij}^H D_{ij})^{-1}\mathcal{B}_{ij}.$$

(Compare this with (4).) Since these represent signal amplitudes and phases, the FMLE algorithm can be used in place of beamforming in situations where signal powers are required. With $T = 2N$, the coarse search can be performed over $N/2$ points of a $2N$-point

zero-padded DFT and the fine 1-D or 2-D searches using all $2N$ points. The complexity for this implementation of the method assuming one coarse and one fine 2-D search is then $8(N/2)^2 + 8(\alpha 2N)^2 = 8(1/4 + 4\alpha^2)N^2$, where α is the fraction of the total number $2N$ of frequency bins over which the fine search is made. For a search over 16 bins with $N = 64$, $\alpha = (16/128)$ and the total flop count is $8(5/16)N^2$. For N beams the beamforming complexity is $8NT = 16N^2$. Taking the preprocessing into account, the FMLE algorithm is about a factor of 4 faster than beamforming. In Figures 1 and 2, complexities for 32-element and 48-element linear arrays are given. In these examples efficiency has been traded for increased resolution, and the FMLE method as implemented here searches over twice as many DOAs as there are beams, with a resulting decrease in efficiency and increase in resolution.

5. CONCLUSIONS

It has been shown that the DFT/periodogram method used to find the maximum likelihood estimate of a single DOA can be extended to obtain maximum likelihood estimates for multiple DOAs. The generalized DFT/periodogram method is summarized in the following:

$$\arg\max_{\alpha} \ \text{trace} \ \left(\mathcal{B}_\alpha^H (D_\alpha^H D_\alpha)^{-1} \mathcal{B}_\alpha\right). \tag{21}$$

For the case of two sources and one data snapshot, the complexity of the FMLE algorithm is comparable to that of beamforming, with less bias for closely spaced sources. The fundamental reason for the increase in performance of the FMLE algorithm, compared with the standard ML search method, can be seen by comparing equations (5) and (21).

REFERENCES

[1] L.L. Scharf, *Statistical Signal Processing,*. Addison-Wesley, 1991.

[2] D. Luenberger, *Optimization by Vector Space Methods,*. John Wiley and Sons, 1969.

[3] S. Kay, *Modern Spectral Estimation,*. Prentice-Hall, 1988.

[4] G. Golub and C. V. Loan, *Matrix Computations, 3rd Edition,*. Johns Hopkins University Press, 1996.

[5] S. Athans and F. Schweppe, "Gradient Matrices and Matrix Calculations," *MIT Lincoln Laboratory Technical Note 1965-53*, November 1965.

[6] L.V. Ahlfors, *Complex Analysis,* McGraw-Hill, 1966.

APPENDIX

```
%
% del is the separation in DFT bins.  Process all pairs of bins with
% an equal separation at the same time.
% An L-element FFT here.
valmax = -1e10;
for del = 1:L-1
   % inner loop has been vectorized in the following statements.
   phi1=phi(1:L-del);                      % DFT coeff; angle.
   phi2=phi(1+del:L);
   G1 = g(1:L-del);                        % DFT coeff; magnitude squared.
   G2 = g(1+del:L);
   R1 = r(1:L-del);                        % DFT coeff; magnitude.
   R2 = r(1+del:L);
   %
   % R: magnitude of gamma.
   % T: argument of gamma.
   % F: the factor in: F*inv( D'*D).
   %
   dthetax = phi2-phi1+T(del);             % 2 flops per point.
   Rh = (R(del)*2)*cos(dthetax);           % 2 flops per point.
   GG = G1+G2-(R1.*R2.*Rh);                % 4 flops per point.
   %
   [val,index] = max(GG);
   val = F(del)*val;
   if(val>valmax)
      valmax = val;
      iisav = index;
      jjsav = index+del;
      delsav = del;
   end;
end;
```

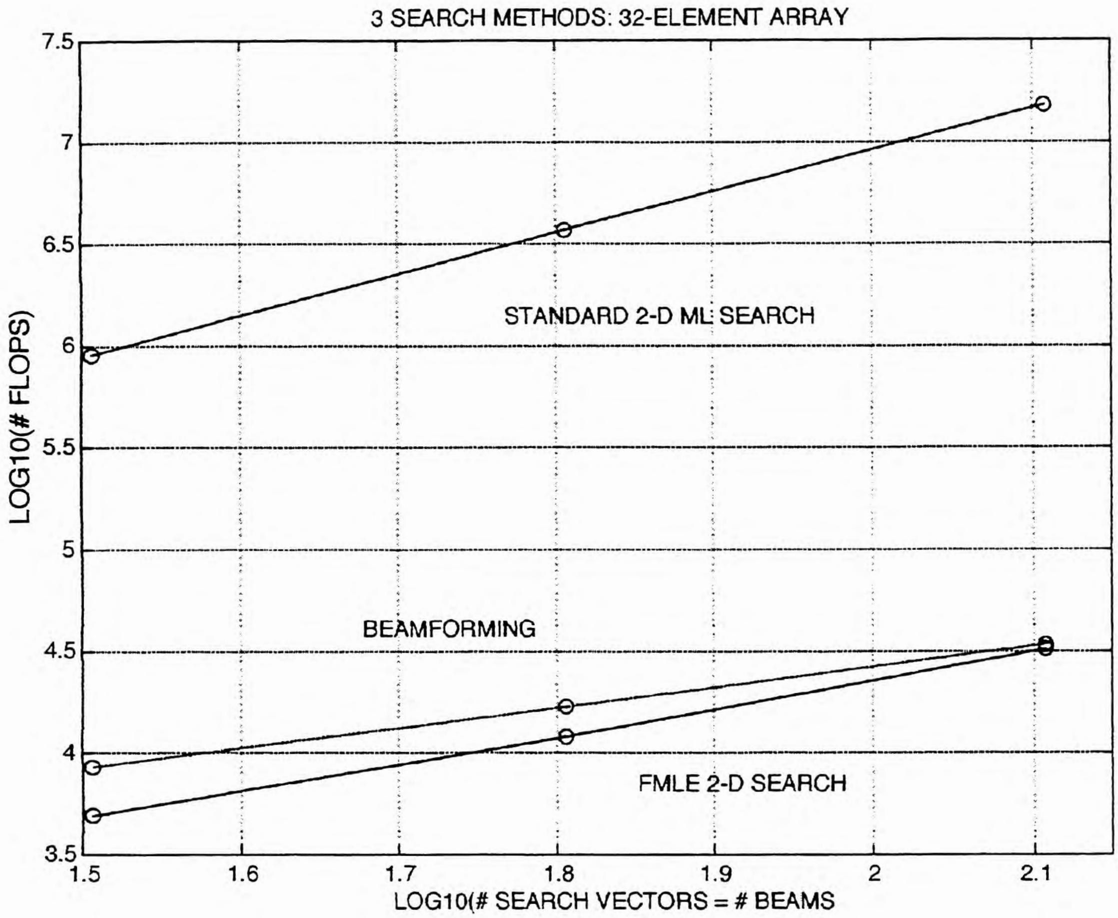

Figure 1. Comparison of 3 Bearing Estimators, 32-Element Array

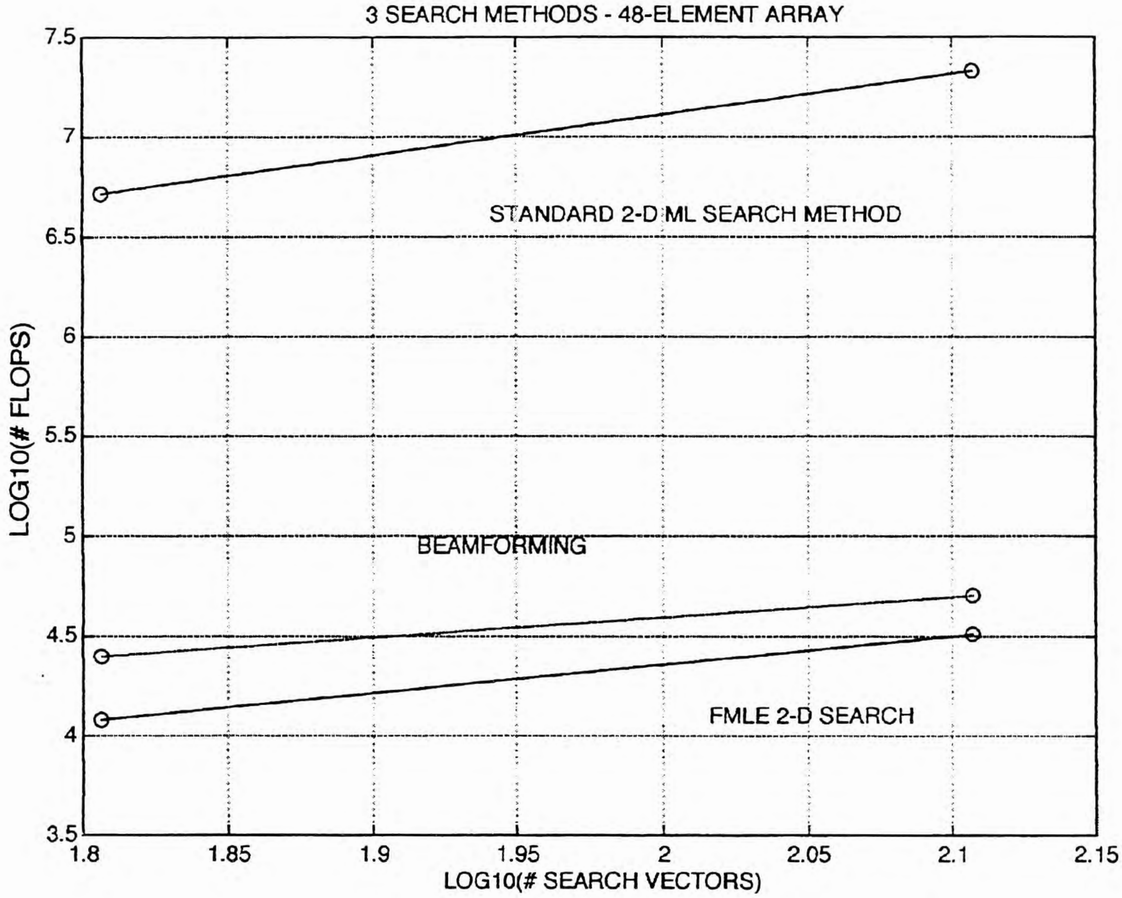

Figure 2. Comparison of 3 Bearing Estimators, 48-Element Array

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